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Fluctuation Induced Almost Invariant Sets

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14. ABSTRACT

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Fluctuation induced almost invariant sets

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We consider the approximation of fluctuation induced almost invariant sets arising from stochastic dynamical systems. We describe the dynamical evolution of densities via the SFP operator. Given a stochastic kernel with a known distribution, approximate almost invariant sets are found by translating the problem into an eigenvalue problem derived from reversible Markov processes. Two examples of the methods are used to illustrate the technique.

I. INTRODUCTION

Transport in dynamical systems is an important topic of active research in that it applies to many problems in physical systems from low to high dimensions. Most of the tools for handling transport from a geometric point view have come from modern dynamical systems, such as [16]. The effects of noise on nonlinear dynamical systems have also become an important topic in recent years. Examples occur in noise induced instabilities arising in deterministic stable dynamics, such as escape from a potential [15], analysis of stochastic bifurcations including exploration of the interaction of noise and global bifurcation based on underlying chaotic saddles [11], noise induced escape using the Hamiltonian theory of large fluctuations [6], the theory of quasi-potentials [9, 10], or a variational formulation of optimal escape paths [13]. It is well known that noise can excite unstable chaotic structures while destroying regular periodic dynamics, but most studies consider noise induced chaos occurring near a bifurcation, such as a saddle-node point or a crisis of chaotic attractors, which leaves a chaotic saddle present.

As an alternative to the above analysis methods, set theoretic methods have been used to quantify the stochastic attractors, as well as transport in stochastic dynamical systems. While for continuous dynamical systems with noise one solves a master equation (such as Fokker-Planck) for the probability density function [15], discrete dynamical systems benefit from using the Frobenius-Perron operator (FP) formalism [12] to determine density evolution. In [4], the idea of examining transition functions for deterministic chaotic systems generated the machinery for discretizing the FP operator, and was applied to kernels which correspond to deterministic systems. Using similar

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ideas, the FP operator was explicitly extended to stochastic kernels in [1], where it was applied to probabilistic transport in epidemic models and discrete low dimensional examples. A full treatment and analysis of the discretization of the FP operator is given in [3], as well as transport from one invariant region to another.

More recently, the transport techniques have been extended to extract almost invariant sets from dynamical systems, sets which are structures in which the dynamics may remain for a long period of time prior to leaving the region. [8] The techniques make use of results from graph theory on reversible, or detailed balance, Markov processes of the probabilistic transition dynamics [7], and we follow a similar approach here for stochastic kernels.

The layout of the remaining part of the paper is as follows. In Section 2 we describe the general theory for the stochastic FP operator. Both continuous and discrete versions are given. Methods for computing approximate almost invariant sets are described. Section 3 has two examples of fluctuation induced almost invariant sets. One is a simple bi-stable one dimensional map and the other is an example from epidemiology. Conclusions are summarized in Section 4.

II. GENERAL THEORY

A. Stochastic Frobenius-Perron operator

We define the Frobenius-Perron (FP) operator for a deterministic function the following way. Let F be map for a discrete dynamical system acting on a set $M \subset R^n$ defined by

$$F : M \to M, \quad x \mapsto F(x). \tag{1}$$

We define an associated dynamical system over the space of densities of ensembles of initial conditions,

$$P_F : L^1(M) \to L^1(M), \quad \rho(x) \mapsto P_F[\rho(x)]. \tag{2}$$

This FP operator (P_F) is defined by [12], [17]

$$P_F[\rho(x)] = \int_M \delta(x - F(y))\rho(y)dy, \tag{3}$$

acting on probability density functions $\rho \in L^1(M)$.

Now consider the stochastically perturbed dynamical system $F_\nu : R^n \to R^n$, $x \mapsto F(x) + \eta$, where η is a random variable having probability density function $\nu(x)$. The stochastic Frobenius-

Perron (SFP) operator is defined to be

$$P_F[\rho(x)] = \int_M \nu(x - F(y))\rho(y)dy. \tag{4}$$

For our applications, we will use a normal distribution in R^n given by

$$\nu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\|x\|^2/2\sigma^2) \tag{5}$$

with mean $\langle x \rangle = 0$ and standard deviation σ. For a discrete time system with constantly applied stochastic perturbation having normal distribution, it is clear that the SFP operator is

$$P_F[\rho(x)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_M e^{-\frac{\|(x-F(y))\|^2}{2\sigma^2}} \rho(y)dy. \tag{6}$$

B. The discrete theory

In order to compute finite dimensional approximations to the SFP operator, we approximate functions in $L^1(M)$, with discretely indexed basis functions $\{\phi_i(x)\}_{i=1}^{\infty} \subset L^1(M)$. The finite dimensional linear subspace is generated by a subset of the basis functions, $\Delta_N = span(\{\phi_i(x)\}_{i=1}^{N})$, such that $\phi_i \in L^1(M)\forall i$. One approximates density $\rho(x)$ by the finite sum of basis functions, $\rho(x) \simeq \sum_{i=1}^{N} c_i\phi_i(x)$, where $\phi_i(x) = \chi_{B_i}(x)$, χ_B is an indicator defined on boxes $\{B_i\}$ covering M. The infinite-dimensional matrix is approximated by the $N \times N$ matrix,

$$A(B_i, B_j) \equiv A_{i,j} = (P_{F_\sigma}[\phi_j], \phi_i) = \int_M P_{F_\sigma}[\phi_j(x)]\phi_i(x)dx \tag{7}$$

for $1 \leq i, j \leq N$. Therefore, a transport matrix entry $A_{i,j}$ value represents how mass flows from cell B_i to cell B_j. Normalizing $A_{i,j}$ by the measure of B_i yields the probability that a point in B_i has its image in B_j, which is denoted by $P_{i,j}$. P is therefore a stochastic, or probability, transition matrix.

Almost invariance can now be considered in terms of a given partition which covers the set M. We can think of a set being almost invariant if its self transition $A(B_i, B_i)$ is close to 1. Therefore, we may adopt the following description of almost invariance. We say an almost invariant set consists of partition $\aleph = \{B_1, \cdots, B_M\}$ such that $\mathcal{A}(\aleph) = \frac{1}{M}\sum_1^M A(B_i, B_i)$ is maximized over all partitions. [8] In order to compute the almost invariant sets, we will require some modifications to the transport matrix.

C. Properties of the transport matrix

Given a stochastic transition matrix P where rows sum to unity, there are several quantities that can be easily computed from the Perron-Frobenius theorem:

1. Probabilities evolve discretely according to $\pi^{(n+1)} = P^T \pi^{(n)}$.

2. There exists a left eigenvector π of P which approximates the invariant probability density of the dynamical system. It satisfies $\pi^T = \pi^T P$, since it has an eigenvalue of 1.

To consider almost invariant sets of Markov chains, we will need to create a transition matrix that is reversible. The idea of using reversible Markov processes to find almost invariant sets comes from an algorithm in [7], which in turn uses results derived from ergodic graph theory ideas applied to image segmentation. [14]

Let P be a stochastic primitive matrix. [18] For such a matrix, there exists a unique probability vector, such that it is a left eigenvector with eigenvalue 1; i.e., $\pi^T = \pi^T P$. It also has a right eigenvector, $\mathbf{1} = (1, \ldots, 1)^T$. Since $\sum \pi_i = 1$, the left and right corresponding eigenvectors satisfy $\pi^T \mathbf{1} = 1$. The interpretation of π is very useful, since each component π_i represents the fraction of time the dynamics spends in box B_i.

We now consider the idea of a reversible Markov chain, or detailed balance. A reversible Markov chain is one which satisfies $\pi_i P_{ij} = \pi_j P_{ji}$ for all i, j. Such a condition implies π is an invariant distribution of P. A reversible Markov chain in equilibrium is one whose forward sequence of events has the same probability of the reverse sequence, making it difficult to state which direction time is going in a real experiment. Letting $D = diag(\sqrt{\pi}_j)$, it is easy to show detailed balance is equivalent to $D^2 P = P^T D^2$.

Since in general P is not reversible as we have defined it from Eq. 7, we construct a reversible Markov from our transition matrix. Let $R = (P + \hat{P})/2$, where $\hat{P}_{ij} = \frac{\pi_j}{\pi_i} P_{ji}$ and π is an invariant probability density of P. Then the following properties may be shown to be true:

1. $D^2 R = R^T D^2$, so R is reversible.

2. $\sum_{j=1}^{n} R_{ij} = 1$.

3. $\pi^T R = \pi^T$.

4. If R is reversible, the eigenvalues are real and the eigenvectors are orthogonal.

Property 4 is especially important, since it will used to cluster the invariant sets.

In order to compute a collection of sets which are almost invariant, one may examine eigenvectors of R, which reveal the approximate almost invariant partitions. Our machinery based on the SFP operator defined in Eq. 4 allows us to do this quite easily. The basic idea behind generating the invariant sets via the eigenstructure of R may be understood as follows.

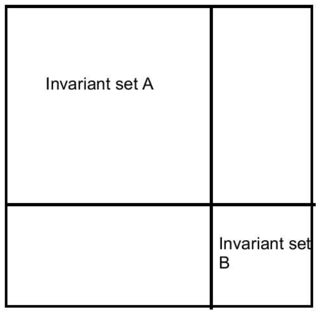

FIG. 1: invariant set sketch

Consider the deterministic case of Eq. 1 in the absence of noise, and assume there exist two basins of attraction, A and B. Each of these sets are made up of aggregates of the appropriate union of sets from the partition $\aleph$. Given any non-empty index set I, define its characteristic vector $\mathbf{1}_I$ by $\mathbf{1}_{I,i} = 1$ for all $i \epsilon I$, and zero otherwise. Then the one step transition probability from a set A to B is given by

$$w_\pi(A, B) = \frac{\sum_{a\in A, b\in B} \pi_a P_{ab}}{\sum_{a\in A} \pi_a} = \frac{< \mathbf{1}_B, P\mathbf{1}_A >_\pi}{< \mathbf{1}_A, \mathbf{1}_A >_\pi}, \tag{8}$$

where $<, >_\pi$ is an inner product weighted by the distribution π; $< x, y >_\pi = x^T D^2 y$. (This should be compared with Eq. 7.) Clearly, if A is an invariant for the deterministic case, $w_\pi(A, A) = 1$. Using the FP machinery, it is clear that if partition $\aleph$ may be decomposed into two disjoint invariant sets, then the stochastic case should be a perturbation of these sets [5]. In the deterministic case, if A and B are invariant and uncoupled, then there exist right eigenvectors of R corresponding to each diagonal block having eigenvalue of 1, with eigenvector entries equal to ones. That is, there exists an eigenspace spanned by the vectors

$$\chi_A = (\mathbf{1}_A^T, 0) \quad \text{and} \quad \chi_B = (0, \mathbf{1}_B^T). \tag{9}$$

Using Eq. 9, we can write any basis of the eigenspace in terms of eigenvectors which are constant on the invariant sets. Perturbation theory states that we expect this structure to persist in the stochastic cases as well, where now the sets A and B are almost invariant in the diagonal blocks in Fig. 1, while the off diagonal blocks represent sets in which transition occur of the form $A \to B$, and vice versa. In [1], we have exploited this idea using the graph theoretic notions presented in detail in [3].

Once we have the eigenstructure of R, we may associate different signs with different eigenstructures. That is, we find the first few eigenvalues of R clustered near unity, and examine the

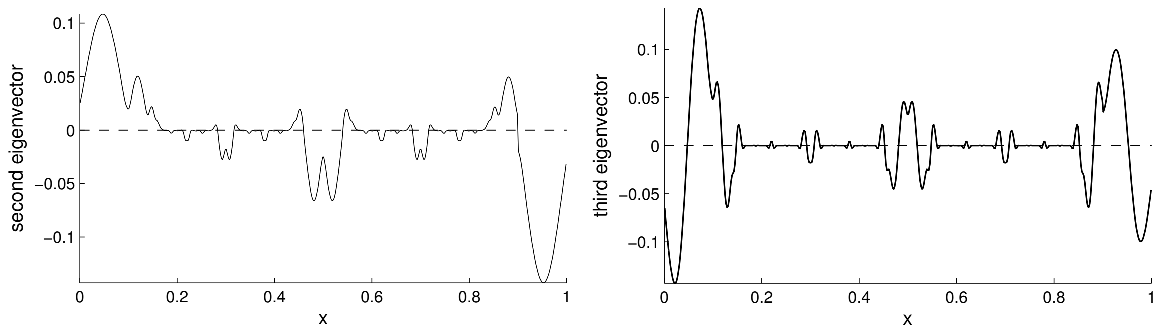

FIG. 2:

Plots of the second and third eigenvectors (left and right) of the stochastic one dimensional map given in Eq. 10 with a standard deviation of $\sigma = 0.005$.

eigenstates, which will be a mixture of almost invariant sets formed from the above basis elements. (In [7], they use clustering arguments based on fuzzy set theory, which we do not do here.) Since the first eigenvector of R is a vector of ones, and Property 4 states that the second eigenvector must be orthogonal to the first, we expect the second eigenvector to see if it contains almost invariant sets emanating from two basins of attraction. Therefore, we may expect to use a signed structure to associate the right eigenvectors of a reversible matrix with each almost invariant set. (There do exist other approaches, but these will be filled in the discussion.) We illustrate this with two examples, both of which consist of bi-stable invariant attractors in the deterministic cases.

III. EXAMPLES

In this section we present two examples: one a discrete map, and another a driven flow perturbed with a discrete noise source.

A. A one dimensional map

The one-dimensional piecewise linear map, $f : [0, 1] \to [0, 1]$,

$$f(x) = \begin{Bmatrix} -0.9x + 0.09 & \text{if} & x < 0.1, \\ 2.5x - 0.25 & \text{if} & 0.1 \leq x < 0.5, \\ -2.5x + 2.25 & \text{if} & 0.5 \leq x < 0.9, \\ -0.9x + 1.81 & \text{if} & 0.9 \leq x, \end{Bmatrix} \tag{10}$$

There are two stable fixed points, $x = 9/19$ and $x = 18.1/19$, with trapping regions $[0, 0.1]$ and $[0.9, 1]$, respectively. There are also two unstable fixed points, $x = 1/6$ and $9/14$. This is a one-

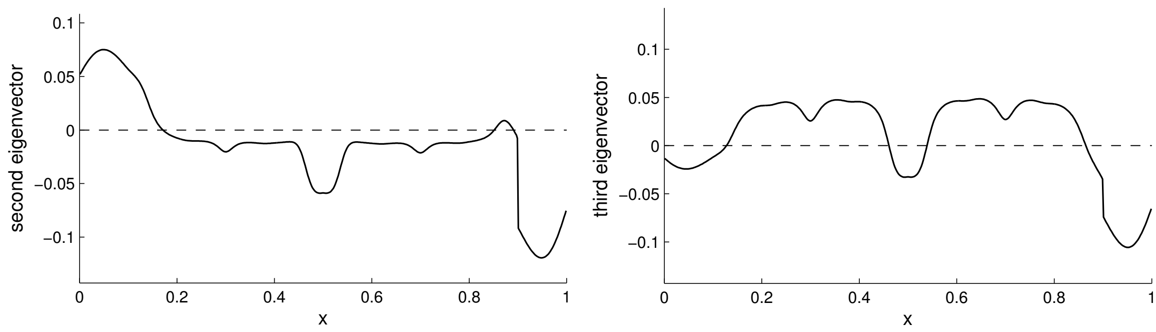

FIG. 3:

Plots of the second and third eigenvectors (left and right) of the stochastic one dimensional map given in Eq. 10 with a standard deviation of $\sigma = 0.05$. Notice that the small noise case in Fig. 2 picks up almost invariant sets with more detail than the larger noise case here.

dimensional bi-stable system. Note that in this example, we know that each of the two basins forms a Cantor set in $[0, 1]$, with a naturally finer and finer scale associated with each successive pre-image of the trapping region. Given such a structure, it is clear that the basins are intertwined, as shown in [3]. The ordering of the sets is therefore quite easy to determine so that a block diagonal matrix may be constructed for invariant measures.

In Figs. 2 and 3, we have implemented the SFP operator in Eq. 4 with the normal distribution on the map given by Eq. 10. Using the invariant distribution, we computed a reversible Markov process. The first right eigenvector was found to be uniform, as expected, and the second and third eigenvectors are plotted. The signs of the entries in the second eigenvectors denote the almost invariant sets for two distinct noise sources. Figure 2 shows the eigenvectors corresponding to a standard deviation of $\sigma = 0.005$, and Fig. 3 has a higher standard deviation of $\sigma = 0.05$. One interesting observation in the second eigenvector for smaller noise is that the sets which are the union of intervals which are positive appear to come from the attractor on the left, while the negative values come from the attractor on the right. In the large noise case, it appears that the almost invariant set becomes smoother, since the standard deviation used in the SFP kernel is an order of magnitude larger.

B. Predicting almost invariant sets from epidemics

We consider a modified epidemic model for the spread of disease where the state variables are susceptibles S and infectives I. (See [2] for a derivation and notation of parameters.) The modified

SI model (MSI), given by:

$$
\begin{aligned}
S'(t) &= \mu - \mu S(t) - \beta(t)I(t)S(t) \\
I'(t) &= \left(\frac{\alpha}{\mu+\gamma}\right)\beta(t)I(t)S(t) - (\mu+\alpha)I(t). \\
\beta(t) &= \beta_0(1+\delta cos 2\pi t)
\end{aligned}
\tag{11}
$$

The parameter values are set to the standard used for measles, and are fixed at $\mu = 0.02$, $\alpha = 1/0.0279$, $\gamma = 1/0.01$, $\beta_0 = 1575.0$, $\delta = 0.1$.

The solutions of the MSI model as well as the bifurcation diagrams for the above set of parameters agree quite well with those of the full SEIR model. For a full description of the deterministic solutions, see [2]. Since the MSI model is periodically driven with period one and both S and I are fractions of the population, it may be a viewed as a two-dimensional map of the unit box into itself. The stochastic model is considered to be discrete as well for the purposes of this paper. That is, noise is added to the population rate equations periodically (period=1) at the same phase having mean zero and standard deviation σ. The dynamics may then be represented as a map, $F : \Re^2 \to \Re^2$, where

$$
(S, I)(t+1) = F[(S, I)(t)] + \eta(t).
$$

Here $\eta(t)$ is a discrete noise term.

In [2], it was shown that the existence of bi-instability is a sufficient condition for noise to excite chaos. That is, there exists a sufficiently large standard deviation in which new unstable orbits are created and sampled. For the parameters listed, in the zero noise case there exist period one and period three attractors. The transport due to the addition of noise using Eq. 4 was originally done in [1, 3]. Here we extend the results to extract the almost invariant sets by analyzing the eigenstructure of the second and third vectors of R.

We show the small noise almost invariant sets ($\sigma = 0.005$) in Fig. 4, while larger noise ($\sigma = 0.035$) results are seen in Fig. 5. Notice that in the left panel of Fig. 4, there are two almost invariant sets with opposite signs. The red regions correspond to sets which are near the deterministic basin boundary for the period 3 attractor, while the dark blue correspond to sets near a period 2 attractor. The third eigenvector begins to show something new. The red and yellow regions correspond to the almost invariant sets contained in the second eigenvector, but the dark blue region points to an almost invariant set of an unstable region. This region is what we would call a fluctuation induced almost invariant set.

Further examination of the fluctuation induced almost invariant set may be made by considering a larger noise amplitude. Intuitively, we expect the noise to allow the dynamics to sample larger

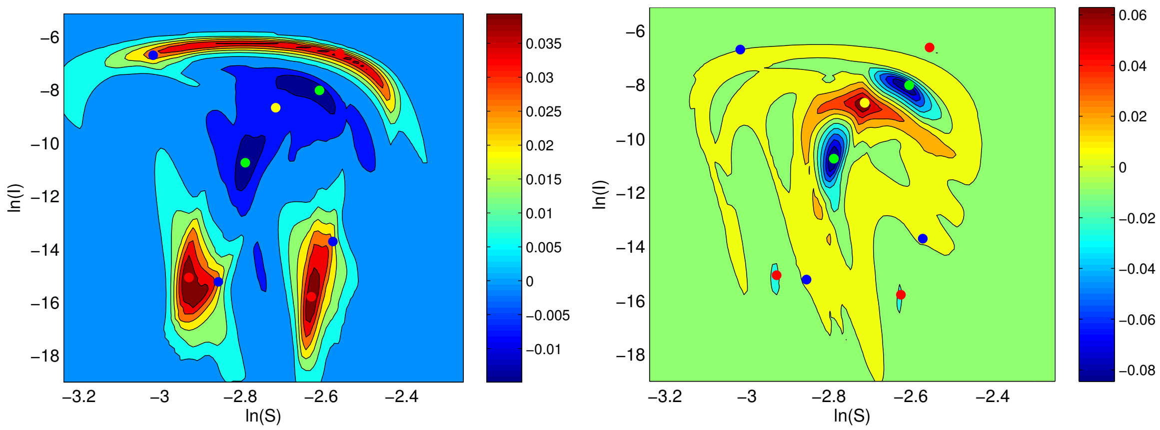

FIG. 4: The plots show a projection of the second and third eigenvectors of the reversible matrix R for the MSI model in Eq. 11. The parameters are listed in the text. The standard deviation is $\sigma = 0.005$, which is a small noise case.

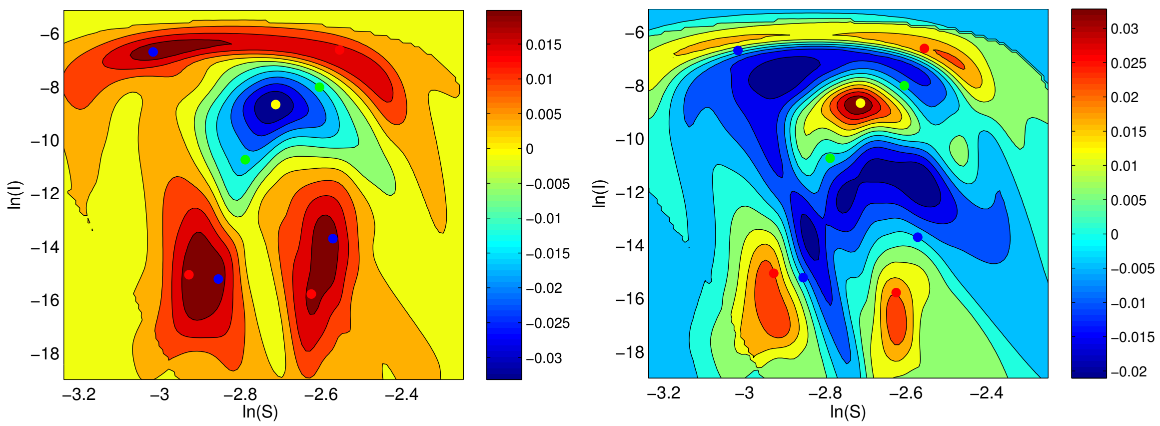

FIG. 5: Same as in Fig. 4, but with large noise standard deviation, $\sigma = 0.035$.

regions of phase space. The current machinery allows us to describe more fully how these regions behave. Figure 5 clearly shows that the effects of noise enlarge the almost invariant regions. In the left panel, the period 3 regions are still distinct, but now the period two region looks like a large region centered around a period unstable point. In the right panel, it is clear that the fluctuation induced region is enlarged considerably, and it is expected that the dynamics spends a good deal of time in this region.

IV. CONCLUSIONS

We considered the approximation of fluctuation induced almost invariant sets arising from stochastic dynamical systems. We showed that by starting with the stochastic FP operator, a

discrete transition matrix may be computed in one step, the problem of formulating an approximation to the almost invariant sets could be translated into finding the eigenstructure of a reversible Markov process. The actual methods for locating clusters, or aggregates of sets, have its roots in many problems, and therefore, there are many alternative techniques. The methods used in [7, 8] find their origins in image segmentation. [14] In those works, as here, we based our cluster analysis on signed eigenstates of a reversible Markov process. However, other techniques such as fuzzy set theory, greedy algorithms, and graph theory are all methods which may be implemented to formulate a cluster analysis algorithm.

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[17] The other common form of the Frobenius-Perron operator being, $P_F[\rho(x)] = \sum_{y:y=F^{-1}(x)} \frac{\rho(y)}{|F'(y)|}$, where the sum is taken over all pre-images of F.

[18] A matrix is called primitive if there is an integer $k > 0$ such that $P^k > 0$.